

The Search for the Primitive

1 Introduction

In a recent interesting paper by Gluchoff, [G], the development of the theory of integration was related to the needs of trigonometric series. The paper ended with the introduction of the Lebesgue¹ integral at the beginning of the century.

More usually the case is made for the the development of the integral being influenced by the problem of finding anti-derivatives, or primitives. Again the story could end with the integral of Lebesgue, as this is the integral of everyday mathematics. The needs of the search for a primitive, as Lebesgue himself called it, [L], had produced a tool of first-rate importance for the whole of mathematical analysis. In neither of these stories was the original goal attained; the Lebesgue integral neither calculated the coefficients of all trigonometric series, nor gave the primitives of all derivatives. However interest in the two problems waned because of the overwhelming importance of the Lebesgue integral. Also, it was a tool of great power in the study of Fourier series, and did find all but the most obscure primitives.

The question of completing the story for trigonometric series will be taken up in another paper, [Bu2]. Here we will study how the the search for a primitive led to the Lebesgue integral, show how the search ended within a decade of the introduction of this integral, and discuss work that continues to this day, more than three centuries after the basic work of Newton².

This topic is better covered in the literature than the one discussed by Gluchoff so not all details will be given; see for example [H; L; P]. In addition the various entries in [E] are useful, and further references are given there.

Some technicalities are placed in the Appendix, Section 8. Terms used which are explained there will be written as `closed` when they first occur.

2 The Problem

A problem that was the source of much research into integration is the following.

THE CLASSICAL PRIMITIVE PROBLEM *Given that function is a derivative on a closed bounded interval find the function of which it is the derivative.*

This problem is well posed, as the following simple corollary of the *Mean Value Theorem of Differentiation* shows; see [R, p.163].

UNIQUENESS THEOREM *If $F' = 0$ on $[a, b]$ then F is a constant function.*

¹ Henri Léon Lebesgue, 1875–1941.

² Sir Isaac Newton, 1642–1727.

A function F such that $F' = f$ is called a *primitive*, or an *anti-derivative*, of f , and the Uniqueness Theorem shows that although a derivative can have many primitives, any two such primitives differ by a constant.

If then f is a derivative on the interval $[a, b]$ it has a unique primitive \tilde{F} satisfying $\tilde{F}(a) = 0$; this we will call *the primitive of f* , or *the anti-derivative of f* . Equivalently if F is a primitive of f then the primitive of f is,

$$\tilde{F}(x) = F(x) - F(a), \quad a \leq x \leq b.$$

3 The Newton Integral

Originally there was no primitive problem since all functions were derivatives, and every function had a derivative, except at the most obvious points—like $x^{2/3}$ at the origin—and finding the primitive of a derivative was solved by a theorem of Newton.

THE FUNDAMENTAL THEOREM OF CALCULUS *Given a function on the interval $[a, b]$ it is the derivative of the area under its graph*

If the area under the graph of f and above the interval $[a, x]$ is $A(x)$, then this theorem says that $A' = f$; since in addition $A(a) = 0$ it follows that A is the primitive of f and the classical primitive problem is solved.

There is a minor problem: by talking of the area under the graph we seem to be assuming f is positive. This can be taken care of, as in any elementary calculus course, by understanding area to mean the area above the axis minus the area below the axis.

Another problem was the concept of area; it was not defined, being considered as an obvious idea.

This was the state of the subject for some time after Newton, and is the approach that is common in almost all elementary calculus courses today. The quantity A is sometimes called the *Newton integral of f* .

The classical primitive problem arose because of two developments.

- (I) A proper definition of area was given, and the definition was not related to the definition of derivative.
- (II) The concept of function broadened, and with this came the realization that a derivative can be badly behaved; it can fail to be continuous, and it can be unbounded, at many points

These developments meant that now there are two classes of functions; one contains all functions which have areas under their graphs, and one contains all derivatives. These two classes can be distinct, and even if a function is in both of these classes we must still see if the Fundamental Theorem of Calculus holds with “*Given a function...*” replaced by “*Given a function that is a derivative...*”

In more common usage: we define an integral (the area), and so the class of integrable functions; are derivatives then integrable, and if so is the indefinite integral a primitive? Is the derivative of the indefinite integral the original function that was integrated?

Unfortunately the answer to the first these questions was essentially no. The integral introduced by Cauchy³ and Riemann⁴ was better than the Newton integral, it defined area properly and the class of integrable functions was large. Unfortunately, although many non-derivatives could be integrated, not all derivatives were integrable.

During the half century following the basic work by Cauchy and Riemann, the search for an integral that would solve the classical primitive problem was pursued. However for a solution several tools had to be in place:

- (a) a full understanding of the nature of sets;
- (b) a good definition of measure, or area, of sets;
- (c) a deeper knowledge of the properties of a derivative.

These first two requirements were met by the beginning of the twentieth century following the work of Cantor⁵, Baire⁶, Borel⁷ and Lebesgue. The required properties of derivatives were obtained soon after by Denjoy⁸, who then gave the first full solution of the problem.

4 The Integrals of Cauchy and Riemann

4.1 THE RIEMANN INTEGRAL Cauchy laid the foundations for most of analysis and in particular he gave the first proper definition of area as an integral, for continuous functions; and he proved the Fundamental Theorem of Calculus for his definition.

THEOREM 1 *If a function is a continuous derivative then it has an area under its graph and this area is the primitive of the function.*

The definition of area given by Cauchy was a very methodical form of Archimedes⁹ *method of exhaustion*, [Gr, pp.29–30]. Look at the region under graph and divide it into n slices and approximate the area of each slice by a rectangle of height the value of the function at the left-hand side of the slice, or in fact at any point in the

³ Augustin-Louis Cauchy, 1789–1857.

⁴ Georg Friedrich Bernhard Riemann, 1826–1866.

⁵ Georg Ferdinand Ludwig Cantor, 1845–1918.

⁶ René-Louis Baire, 1874–1932.

⁷ Emile Félix-Edouard-Justin Borel, 1871–1956.

⁸ Arnaud Denjoy, 1884 –1974.

⁹ Archimedes, 287(?)BC–212 BC.

slice; see [P, p.6]. If as the width of the largest slice gets small the total area of all n rectangles has a limit, this says that there is an area under the graph and gives the value of this limit as the value of the area; see [Gr, pp.140–148; H, pp.9–11; P, pp.3–4].

Shortly after this Riemann gave the modern form of the definition. His definition made no assumption about the nature of the function being integrated, the class integrable functions, the class of functions for which an area existed, was precisely the class for which the calculation worked.

Importantly, Riemann gave a fundamental necessary and sufficient condition for a function to be integrable.

Let us give Riemann's definition as, although it is well known, see [K, Chap.II; R, Chap.VI], we will need it for a later development.

Given a closed interval $[a, b]$ then a *partition* of that interval is a finite set of points, $\varpi = \{a_0, \dots, a_n\}$ say, such that $a = a_0 < a_1 < \dots < a_n = b$.

The *norm of a partition*, $||\varpi||$, is the length of the longest interval of the partition, that is $||\varpi|| = \max_{1 \leq j \leq n} |a_j - a_{j-1}|$.

A *tagged partition* of $[a, b]$ is a partition ϖ , as above, together with a *tag* in each interval, that is an ξ_j , $a_{j-1} \leq \xi_j \leq a_j$, $1 \leq j \leq n$. A tagged partition will be written $\varpi = \{a_0, \dots, a_n; \xi_1, \dots, \xi_n\}$.

If f is a function defined on $[a, b]$ and if ϖ is a tagged partition of $[a, b]$, form the *Riemann sum*

$$S = S(f; \varpi) = \sum_{j=1}^n f(\xi_j)(a_j - a_{j-1}). \quad (1)$$

If there is a real number I such that given any $\epsilon > 0$, there is a $\delta > 0$ and for all tagged partitions ϖ of norm less than δ we have that $|S - I| < \epsilon$ then f , the *integrand*, is said to be *Riemann, \mathcal{R} -, integrable on $[a, b]$* with *integral* equal to I ; this written,

$$I = \mathcal{R}\text{-}\int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{j=1}^n f(\xi_j)(a_j - a_{j-1}). \quad (2)$$

The right-hand side of (2) is an abuse of notation, in that it is not sufficient to let n , the number of points in the partition, tend to infinity; we must require that $||\varpi||$ tend to zero, when in particular, $n \rightarrow \infty$.

The indefinite \mathcal{R} -integral is called an *\mathcal{R} -primitive*, and if it is zero at the left-hand endpoint of the interval, *the \mathcal{R} -primitive*.

It is important to notice further that all choices of tags must be allowed, and this has a restrictive implication.

For the \mathcal{R} -integral of f to exist f must be bounded.

The basic properties of this integral are well known and are developed in all books on analysis; see for instance [R, pp.184–203]. In particular if f is continuous, or if f is monotonic, or if f is continuous with a finite number of discontinuities then f is integrable; see [R, pp.192–193, 196].

The important result of Riemann mentioned above tells us just when a function is \mathcal{R} -integrable.

THEOREM 1 *A bounded function is Riemann integrable if and only if it is continuous almost everywhere.*

It is immediate from this result and the examples discussed in 8.4 that not all derivatives are Riemann integrable.

Derivatives need be neither bounded nor continuous almost everywhere. However we have the Fundamental Theorem of Calculus for this integral, due to Darboux¹⁰; [H, pp.50–51; Ho, pp.484–486; K, pp.46–50; L, p.88]

THEOREM 2 *If a derivative is \mathcal{R} -integrable then the \mathcal{R} -primitive is the primitive, that is $(\mathcal{R}\text{-}\int_a^x f)' = f(x)$, $a \leq x \leq b$.*

4.2 THE CAUCHY- RIEMANN INTEGRAL Suppose that a function f is continuous, or just continuous almost everywhere, but not bounded on any interval containing a point x , then we will call x a *point of unboundedness of f* .

EXAMPLES (i) The functions $f(x) = 1/\sqrt{x}$, or $g(x) = \csc 1/x$ have the origin as a point of unboundedness. In the second case the origin is also the limit of a sequence of points of unboundedness; it is a **limit point** of the set of points of unboundedness. In both cases the function can be defined arbitrarily at the points of unboundedness.

Let S be the set of all points of unboundedness of f . Then S is a **closed set**, and the function is not \mathcal{R} -integrable on any interval that meets S , but is integrable on any interval that does not meet S .

If S is empty then the function is \mathcal{R} -integrable, and over the next century various integrals were defined that allowed for more and more complicated types of non-empty sets S ; see [P, p.10].

Cauchy himself had already introduced an extension of his integral method that allows us to handle the case where S is finite.. This method is well known from elementary calculus where it goes under the name of the *improper or infinite integral*; see [K, pp.54–55; R, pp.221–226].

Let us first assume that S consists just of the point a , the left-hand endpoint of the interval. Then f is not \mathcal{R} -integrable in $[a, b]$ but is on every $[\alpha, b]$, $a < \alpha \leq b$. The

¹⁰ Jean Gaston Darboux, 1842–1917.

Cauchy-Riemann, \mathcal{CR} -, integral of f on $[a, b]$ is

$$I = \mathcal{CR}\text{-}\int_a^b f = \lim_{\alpha \rightarrow a} \mathcal{R}\text{-}\int_{\alpha}^b f,$$

provided the limit exists.

Now if f is a derivative that is continuous almost everywhere on $[a, b]$, bounded on every subinterval $[\alpha, b]$, $a < \alpha < b$, but unbounded at a , see 8.4 Examples (i) with $\alpha = \beta = 2$ say, then it is \mathcal{CR} -integrable, and its primitive is the \mathcal{CR} -primitive. For suppose F is a primitive of f , by Theorem 2

$$\mathcal{R}\text{-}\int_{\alpha}^x f = F(x) - F(\alpha);$$

so, as F is continuous,

$$F(x) - F(a) = \lim_{\alpha \rightarrow a} (F(x) - F(\alpha)) = \lim_{\alpha \rightarrow a} \mathcal{R}\text{-}\int_{\alpha}^x f = \mathcal{CR}\text{-}\int_a^x f.$$

The same techniques apply if the single point of unboundedness is b .

If the single point of unboundedness is c , $a < c < b$ then the above definitions will give $\mathcal{CR}\text{-}\int_a^c f$, and $\mathcal{CR}\text{-}\int_c^b f$, and it is then natural to define

$$\mathcal{CR}\text{-}\int_a^b f = \mathcal{CR}\text{-}\int_a^c f + \mathcal{CR}\text{-}\int_c^b f.$$

Further, it is easy to check in these cases that if f is a derivative that is continuous almost everywhere, with one point of unboundedness then f is \mathcal{CR} -integrable, and the \mathcal{CR} -primitive is a primitive.

This procedure, and the result, is readily extended to the case of S being finite; say for simplicity that the points S form a partition ϖ of $[a, b]$, $\varpi = \{s_1, s_2, \dots, s_n\}$ then f is integrable on each $[\sigma_i, \tau_i]$, $s_i < \sigma_i < \tau_i < s_{i+1}$, $1 \leq i \leq n-1$ and the Cauchy-Riemann integral of f is

$$\mathcal{CR}\text{-}\int_a^b f = \sum_{i=1}^n \lim_{\substack{\sigma_i \rightarrow s_{i-1} \\ \tau_i \rightarrow s_i}} \mathcal{R}\text{-}\int_{\sigma_i}^{\tau_i} f,$$

provided all of the limits exist.

So if f is a derivative that is continuous almost everywhere, and has a finite number of points of unboundedness then its primitive is given by the \mathcal{CR} -primitive; that is the Fundamental Theorem of Calculus, Theorem 2, holds if \mathcal{CR} replaces \mathcal{R} at various points in the statement of that theorem.

Finally let us note the following alternative approach to the \mathcal{CR} -integral.

THEOREM 3 Suppose f is continuous almost everywhere and has a finite set, S , of points of unboundedness. Then f is \mathcal{CR} -integrable if and only if there is a continuous F such that on any $[\alpha, \beta]$ not containing any points of S , $F(\beta) - F(\alpha) = \mathcal{R}\text{-}\int_{\alpha}^{\beta} f$. Then F is a \mathcal{CR} -primitive of f and $\mathcal{CR}\text{-}\int_a^b f = F(b) - F(a)$. In particular f is \mathcal{CR} -integrable if f is a derivative, and then F a primitive of f .

Theorem 3 gives a so-called *descriptive* definition of the \mathcal{CR} -integral, as opposed to the *constructive* definition developed above; this approach seems to have originated with Dirichlet¹¹; see [P, p.12].

There is an important distinction between the Riemann integral and the Cauchy-Riemann integral. Clearly $|f|$ is bounded and continuous almost everywhere when f has these properties, hence $|f|$ is \mathcal{R} -integrable if f is. However this is not a property of the Cauchy-Riemann integral.

EXAMPLES (i) Let us consider a simple modification of the example mentioned above, 8.4 Examples (i) with $\alpha = \beta = 2$; $f = F'$ where

$$F(x) = \begin{cases} x^2 \sin \frac{\pi}{2x^2}, & \text{if } 0 < x \leq 1, \\ 0, & x = 0, \end{cases}$$

Then f consists of two parts one of which is continuous and bounded and the other is $\phi(x) = \frac{\pi}{x} \cos \frac{\pi}{2x^2}$. This graph of ϕ has, as we approach the origin, an infinite number of areas above the axis, a_2, a_4, \dots say, and an infinite number of areas below the axis a_1, a_3, \dots say. Then it can be checked that

$$\mathcal{R}\text{-}\int_{1/\sqrt{2n+1}}^1 \phi = \sum_{k=1}^n (-1)^k a_k, \text{ and so } \mathcal{CR}\text{-}\int_0^1 \phi = \sum_{k=1}^{\infty} (-1)^k a_k$$

However the series is a conditionally convergent; the series $\sum_{k=1}^{\infty} a_k$, is not convergent; so $|\phi|$ is not \mathcal{CR} -integrable; see [K, pp.135–136].

This property is expressed by saying that the Riemann integral is an *absolute integral*, while the Cauchy-Riemann integral is *non-absolute integral*. Clearly then, an integral that solves the primitive problem must be a non-absolute integral.

5 Infinite Sets of Unboundedness

5.1 THE CAUCHY SCALE OF INTEGRALS The above process gives what is called the *Cauchy extension* of the Riemann integral. It is important to note that the process can be applied to any integral. In particular we could replace the basic

¹¹ Johann Peter Gustav Lejeune-Dirichlet, 1805–1859.

integral by the Lebesgue integral. What we want to do is to replace the basic integral by the \mathcal{CR} -integral itself, and so begin an induction that will allow us to handle a derivative with its set of points of unboundedness, S , infinite but countable; that is when S is **reducible**.

Note that to say S is finite equivalent to saying that to saying that, the **derived set**, S' , is empty. The next move would be then to consider the case of S having a finite number of limit points, S' finite, or the **second derived set**, $S^{(2)}$, empty

Assume for simplicity that S has one limit point, and that it is a ; see for instance the function g in 3.2 Examples (i).

Then f is not integrable on $[a, b]$. Any $[\alpha, b]$, $a < \alpha \leq b$, contains only a finite number of points of S and so f could be \mathcal{CR} -integrable on $[\alpha, b]$, as it would be if f were a derivative. If this is the case then the *Cauchy-Riemann integral of order two* of f on $[a, b]$ is

$$I = \mathcal{CR}^{(2)}\text{-}\int_a^b f = \lim_{\alpha \rightarrow a} \mathcal{CR}\text{-}\int_{\alpha}^b f,$$

provided the limit exists.

By an argument analogous to that used in the case of the \mathcal{CR} -integral we can define the $\mathcal{CR}^{(2)}$ -integral when S' is any finite set.

Theorem 3 can be extended to this case:

THEOREM 3₂ *Suppose that f is continuous almost everywhere and the set of points, S , of points of unboundedness has a finite number of limit points. The function f is $\mathcal{CR}^{(2)}$ -integrable if and only if there is a continuous F such that on any $[\alpha, \beta]$ not containing any points of S , and $F(\beta) - F(\alpha) = \mathcal{R}\text{-}\int_{\alpha}^{\beta} f$. Then F is a $\mathcal{CR}^{(2)}$ -primitive of f and $\mathcal{CR}^{(2)}\text{-}\int_a^b f = F(b) - F(a)$. If further if f is a derivative then f is $\mathcal{CR}^{(2)}$ -integrable, and F is a primitive of f .*

It is now an easy induction to extend this to the case when for some n the **derived set of order n** , $S^{(n)}$, is empty. In this way we obtain a sequence of extensions of the Riemann integral, the $\mathcal{CR}^{(n)}$ -integrals, $n = 1, 2, \dots$; where $\mathcal{CR}^{(1)}$ is just \mathcal{CR} . Further if f is a derivative, continuous almost everywhere with $S^{(n)}$ empty it is $\mathcal{CR}^{(n)}$ -integrable and the $\mathcal{CR}^{(n)}$ -primitive is a primitive of f .

Suppose now that no $S^{(n)}$ is empty, then $S^{(\omega)} = \bigcap_{n=1}^{\infty} S^{(n)}$ is also not empty, by the **Cantor Intersection Theorem**, see 8.3.

Assume for simplicity that $S^{(\omega)}$ contains one point, see 8.1 Examples (v), and that this point is a . Then, using the Cantor's Intersection Theorem again, each $[\alpha, b]$, $a < \alpha \leq b$ fails to meet some $S^{(n)}$, $n = n(\alpha)$, and so we can evaluate $\mathcal{CR}^{(n)}\text{-}\int_a^b f$, if this integral exists and then define *Cauchy-Riemann integral of order ω* of f on $[a, b]$

as

$$I = \mathcal{CR}^{(\omega)} - \int_a^b f = \lim_{\alpha \rightarrow a} \mathcal{CR}^{(n)} - \int_\alpha^b f,$$

provided the limit exists.

Proceeding with the induction we get finally a case where the derived set is empty, see 8.1, and the computation has been completed. Call the integral that covers all these cases the $\mathcal{CR}^{(\Omega)}$ -integral, and we have that the Fundamental Theorem of Calculus, Theorem 2, holds if $\mathcal{CR}^{(\Omega)}$ replaces \mathcal{R} at various points in the statement of that theorem.

THEOREM 4 *If a derivative is $\mathcal{CR}^{(\Omega)}$ -integrable, that is if it is continuous almost everywhere with a countable set of points of unboundedness, then the $\mathcal{CR}^{(\Omega)}$ -primitive is the primitive, that is $(\mathcal{CR}^{(\Omega)} - \int_a^x f)' = f(x)$, $a \leq x \leq b$.*

However not all continuous almost everywhere derivative are $\mathcal{CR}^{(\Omega)}$ -integrable; see 8.4 Examples (iii).

A derivative can be continuous almost everywhere and have an uncountable set of points of unboundedness.

5.2 THE HARNACK INTEGRAL If a derivative is continuous almost everywhere and has an uncountable set of points of unboundedness, S , then if $[a_n, b_n]$ is any contiguous interval of the perfect kernel of S , it contains only a countable subset of S . Hence we can, using the $\mathcal{CR}^{(\Omega)}$ -integral, calculate the primitive on $[a_n, b_n]$. Since the derivative is continuous almost everywhere the perfect kernel will be of measure zero, and with the hindsight of the Lebesgue integral we expect this set to contribute nothing to the value of the primitive. In which case we would naturally expect the primitive to be defined by an integral introduced by Harnack¹².

$$I = \mathcal{HR} - \int_a^b f = \sum_{n=1}^{\infty} \mathcal{CR}^{(\Omega)} - \int_{a_n}^{b_n} f. \quad (3)$$

This definition is valid provided the series converge. More precisely, since the ordering of the contiguous intervals is arbitrary we will need the series in (3) to converge absolutely; see [G, p.89; Ho vol.I, p.350; P, p.21].

Unfortunately there are derivatives that are continuous almost everywhere for which the series above do not converge; see [D].

In general we would expect that the *Harnack-Riemann, \mathcal{HR} -, integral* to be defined if (i) the $\mathcal{CR}^{(\Omega)}$ -integral has been computed on all the contiguous intervals

¹² A Harnack, 1851–1888.

$[a_n, b_n], n = 1, 2, \dots$ of a perfect set P , (ii) the series of these integrals converges absolutely, and (iii) we know what we mean by the integral over P and then

$$I = \mathcal{HR}\text{-}\int_a^b f = \mathcal{R}\text{-}\int_P f + \sum_{n=1}^{\infty} \mathcal{CR}^{(\Omega)}\text{-}\int_{a_n}^{b_n} f = \mathcal{R}\text{-}\int_a^b 1_P f + \sum_{n=1}^{\infty} \mathcal{CR}^{(\Omega)}\text{-}\int_{a_n}^{b_n} f,$$

where 1_P is the *indicator function of the set P* , that is

$$1_P(x) = \begin{cases} 1, & \text{if } x \in P, \\ 0, & \text{if } x \notin P. \end{cases}$$

Then we have the following result due to Lebesgue, a Fundamental Theorem of Calculus for the \mathcal{HR} -integral; see [L, pp.209–211].

THEOREM 5 *If the derivative f is \mathcal{HR} -integrable, then the \mathcal{HR} -primitive is the primitive, that is $(\mathcal{HR}\text{-}\int_a^x f)' = f(x)$.*

6 The Lebesgue Integral

It was natural to attempt to give area a direct definition and connect this with the Riemann integral. This was done by Jordan¹³ who defined the *Jordan content* of a set by a method that is based on the idea of upper and lower Riemann sums; see [R, pp.184–188].

This approach was generalized by Lebesgue whose definition of measure for open and closed sets is given in 8.2.

The definition of integral that follows from this different approach is the Lebesgue, \mathcal{L} -, integral, the basic tool of modern analysis. The class of integrable functions is based on a class that is naturally connected to the measure introduced, the class of *measurable functions*. This important vector space of functions includes all continuous functions, and all monotonic functions; and importantly, it is closed to pointwise limits, that is the limits of sequences of measurable functions are also measurable. The Lebesgue integral, is well covered in all analysis texts so will not be given here; see [B-B-T; Go; H; Ho vol.I; K; L; N vol.I, pp.116–184]. A few facts are worth noting for our purpose.

THEOREM 6 (a) *The \mathcal{L} -integral extends the \mathcal{R} -integral; that is to say it integrates every Riemann integrable function f , and then $\mathcal{L}\text{-}\int_a^x f = \mathcal{R}\text{-}\int_a^x f$, $a \leq x \leq b$.*

(b) *A non-negative function is \mathcal{L} -integrable if and only if the area under the graph is measurable; and the measure of this area is equal to the value of the integral.*

¹³ Marie Ennemond Camille Jordan, 1838–1922

- (c) If f is measurable and bounded it is \mathcal{L} -integrable.
- (d) The \mathcal{L} -integral is an absolute integral.
- (e) If $f = g$ almost everywhere then f is \mathcal{L} -integrable if and only if g is \mathcal{L} -integrable.

Because of (d) the \mathcal{L} -integral cannot solve the primitive problem, see the remark at the end of 3.2 and Examples 3.2(i); more generally see 8.4 Examples (i).

However in spite of the last remark the implications of the \mathcal{L} -integral for our problem are significant. Derivatives are measurable, since they are limits of sequences of continuous functions see, 8.4, and the following form of the Fundamental Theorem of Calculus, due to Lebesgue, [L p.174; P, p. 68], is a simple consequence of the bounded convergence theorem; see [H, p. 128].

THEOREM 7 *If a derivative is bounded then it is \mathcal{L} -integrable and the \mathcal{L} -primitive is the primitive, that is $(\mathcal{L}\text{-}\int_a^x f)' = f$, $a \leq x \leq b$.*

In one swoop the primitive problem is solved for all bounded derivatives. In fact Lebesgue shows that the condition of boundedness in Theorem 7, can be replaced by “bounded above” or “bounded below”, see [L, p.183]; so that the only bad points that remain for our problem are those that are unbounded in both directions.

In addition we have the following generalization of Theorem 2, [Ho vol I, pp.596–605; Ru, pp.168–169; L, p.88]

THEOREM 8 *If a derivative is \mathcal{L} -integrable then the \mathcal{L} -primitive is the primitive, that is $(\mathcal{L}\text{-}\int_a^x f)' = f(x)$, $a \leq x \leq b$.*

We can easily handle countable sets of unboundedness by using the Cauchy extension of the Lebesgue integral. Further the Harnack extension can be defined as for the Riemann integral leading to extensions of Theorems 4 and 5; see [L, pp.209–211].

THEOREM 9 (a) *If a derivative has a countable set of points of unboundedness, then it is $\mathcal{CL}^{(\Omega)}$ -integrable, and the $\mathcal{CL}^{(\Omega)}$ -primitive is the primitive, that is $(\mathcal{CL}^{(\Omega)}\text{-}\int_a^x f)' = f(x)$, $a \leq x \leq b$.*

(b) *If the derivative f is \mathcal{L} -integrable on the perfect kernel P of the set S of its points of unboundedness, and if the series $\sum_{n=1}^{\infty} \mathcal{CL}^{(\Omega)}\text{-}\int_{a_n}^{b_n} f$ converges absolutely, where $[a_n, b_n]$, $n = 1, \dots$ are the contiguous intervals of P , that is if f is \mathcal{HL} -integrable, then the \mathcal{HL} -primitive is the primitive, that is $(\mathcal{HL}\text{-}\int_a^x f)' = f(x)$.*

Unfortunately neither of the conditions in (b) need hold so again the primitive problem is not completely solved. This is almost the point that Lebesgue reached in his search for the primitive. He did observe that although the conditions need not hold on the complete interval they always do hold on some sub-interval that meets the perfect kernel P . This crucial observation was the basis of the complete solution given by Denjoy, see 6.1 and [H, p 137]..

There is an other property of the Lebesgue integral that on the one hand gives a descriptive definition analogous to that of the \mathcal{CR} -integral in Theorem 3, and on the other hand provides, and solves, an interesting variant of the classical primitive problem.

A function F is *absolutely continuous* on $[a, b]$ if:

for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$\sum_{k=1}^n (d_k - c_k) < \delta \implies \sum_{k=1}^n |F(d_k) - F(c_k)| < \epsilon, \quad (4)$$

where $\{[c_k, d_k], 1 \leq k \leq n\}$ is any set of non-overlapping sub-intervals of $[a, b]$; that is, the sum of all the $|F(d) - F(c)|$ is to be small whenever the sum is taken over any collection of non-overlapping intervals $[c, d]$ of total small length.

This class of functions is very important; in particular an absolutely continuous function has a derivative almost everywhere, and we have the following extension of the uniqueness theorem.

UNIQUENESS THEOREM₂ If F is AC and $F' = 0$ almost everywhere on $[a, b]$ then F is a constant function.

In addition we have the following theorem that gives a descriptive definition of the \mathcal{L} -integral, and solves the variant of the primitive problem mentioned above.

THEOREM 10 (a) A measurable function f is \mathcal{L} -integrable if and only if there is an absolutely continuous function F such that $F' = f$ almost everywhere; and then $\mathcal{L}\int_a^b f = F(b) - F(a)$.

(b) If f is the derivative almost everywhere of an absolutely continuous function then the primitive of f is the \mathcal{L} -primitive of f .

7 Solutions to the Classical Primitive Problem

7.1 DENJOY'S SOLUTION The first solution of the classical primitive problem was given by Denjoy in 1912 by a method he called *totalization*. The totalization procedure is too complicated to give in detail. No new methods were needed as totalization is a combination of the Lebesgue integral, and its Cauchy and Harnack extensions. What was needed however to make these simple classical tools work was a very exhaustive study of the properties of derivatives. This was done by Denjoy, in a series of very long papers collected together in [D].

Totalization depends on the following facts to get started:

(i) for a derivative the set S of points of discontinuity is a **nowhere-dense** closed set; this means we can start the Cauchy extension on the contiguous intervals of this closed set;

(ii) the set of points at which the series in the Harnack extension diverges is also a nowhere-dense closed set; this means we can also obtain the Harnack extension in lots of places.

As a result, after a countable set of calculations we have a contribution to the total, the integral, on all the contiguous intervals of a nowhere-dense perfect set, P say. However both statements (i) and (ii) hold for every perfect set, and so we can now start again but now using P and get the calculations on the contiguous intervals of a nowhere-dense perfect subset of P . After doing this a countable number of times we arrive, using the **Cantor-Baire Stationary Principle**, see 8.3, at an empty set and the calculations are complete.

By earlier remarks the Denjoy integral is designed to cope with functions that are unbounded in both directions at a large set of points so that Denjoy's totalization procedure is equivalent to obtaining a particular sum of a non-absolutely convergent series of areas. It is a very subtle way of adding together partial sums from the contributions made by the various Cauchy and Harnack extensions in a particular order, chosen to give the correct total for derivatives. The resulting integral is called the *restricted Denjoy integral*, the \mathcal{D}^* -integral, and every derivative is \mathcal{D}^* -integrable, and the \mathcal{D}^* -primitive is a primitive.

THEOREM 11 (a) *The \mathcal{D}^* -integral extends the \mathcal{L} -integral, the $\mathcal{CL}^{(\Omega)}$ -integral, and the \mathcal{HL} -integral*

(b) *The Cauchy and Harnack extensions of the \mathcal{D}^* -integral are equivalent to the \mathcal{D}^* -integral.*

(c) *Every derivative is \mathcal{D}^* -integrable and the primitive of a derivative is the \mathcal{D}^* -primitive.*

This last result (b) means that in some sense we have with this integral reached the end of the road, no further generalization along the lines being pursued are possible.

7.2 PERRON'S SOLUTION Denjoy's solution of the classical primitive problem created a lot of interest at the time and led to other and simpler solutions. The easiest of these other methods was due to Perron¹⁴.

In order to solve the problem of finding a F such that $F' = f$ on $[a, b]$ with $F(a) = 0$, consider the two classes of functions M, m defined by:

$$\underline{D}M \geq f \geq \overline{D}m \quad M(a) = m(a) = 0;$$

(here, as elsewhere, the underbars and overbars indicate lower and upper derivatives respectively).

¹⁴ Oskar Perron, 1880–1975.

These classes are called *major and minor functions of f* respectively. The *Perron integral* is the common value of the $\inf M(b)$, and $\sup m(b)$, if such a common value exists.

It is easy to see that this integral solves the primitive problem since if f is a derivative of F then F is both a major and a minor function and so is the Perron integral of f .

This method of Perron has important applications for differential equations other than the present one, $y' = f$; see [E, vol.7, pp.133–134, Vol.9, p 352; W].

7.3 LUZIN'S SOLUTION Another approach, given by Luzin ¹⁵ within a few months of the Denjoy's announcement of his results, is implicit in Denjoy's work.

The idea is to give a generalization of the concept of absolute continuity so that with this generalization Theorem 10 above holds for the \mathcal{D}^* -integral.

The right generalization of absolute continuity is called *restricted generalized absolute continuity*, ACG^* , which we now explain.

(a) *Firstly the term 'generalized, the 'G' in ACG^* .*

Given a property that holds on a set, then the generalized property holds on a closed interval if the interval is the union of a sequence of sets on each of which the property holds. For example every finite function g on $[a, b]$ is generalized bounded since $[a, b]$ is the union of the sets on which $|g| \leq n$. This simple idea is behind the extension; the Lebesgue inverts bounded derivatives, the \mathcal{D}^* -integral inverts all finite derivatives, that is generalized bounded derivatives; see [Bu1, p.458].

(b) *Now to extend the definition of absolute continuity on an interval to absolute continuity on a set.*

To define AC on E , all that has to be done is to require that the endpoints in (4), $\{c_k, d_k, 1 \leq k \leq n\}$, lie in the set E .

Then we get the class ACG by letting the interval be the union of a sequence of sets on each of which the function is AC .

However this class is too general, as a continuous ACG function need not have a derivative almost everywhere; [Go, p.101; S, p.224].

(c) *Finally the meaning of the term restricted, or the star in ACG^* .*

There is another form of definition (4) in which the quantities $|F(d_k) - F(c_k)|$ are replaced by $\omega(F; [c_k, d_k])$, the **oscillation of the function F on the interval $[c_k, d_k]$** . These two definitions are equivalent on intervals, see [Ho vol.I, pp. 331–337], but a moment's reflection will show that this is not the case for sets, as the use of ω involves values of F off the set E . If the oscillation form is used, we say that a function is absolutely continuous in the narrow or restricted sense, is AC^* , on a set; this is a stronger requirement than AC . Then we get the class ACG^* by

¹⁵ Николай Николаевич Лузин, 1883–1950; also transliterated a Lusin

letting the interval be the union of a sequence of sets on each of which the function is AC^* .

EXAMPLES (i) The function

$$1_{\mathbb{Q}}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0, 1], \\ 0, & \text{if } x \in [0, 1] \setminus \mathbb{Q}, \end{cases}$$

is AC on both sets $\mathbb{Q} \cap [0, 1]$, $[0, 1] \setminus \mathbb{Q}$, being constant on each, and so is ACG on $[0, 1]$; but is not AC^* on either set, and is not ACG^* on $[0, 1]$.

ACG^* -functions have derivatives almost everywhere and we have the following extension of the uniqueness theorem.

UNIQUENESS THEOREM₃ *If F is continuous, ACG^* and $F' = 0$ almost everywhere on $[a, b]$ then F is a constant function.*

In addition we have the following theorem that generalizes Theorem 10, gives a descriptive definition of the \mathcal{D}^* -integral, and gives another variant of the primitive problem.

THEOREM 12 (a) *A measurable function f is \mathcal{D}^* -integrable if and only if there is a continuous ACG^* function F such that $F' = f$ almost everywhere; and then $\mathcal{D}^*\text{-}\int_a^b f = F(b) - F(a)$.*

(b) *If f is the derivative almost everywhere of a continuous ACG^* function then the primitive of f is the \mathcal{D}^* -primitive of f .*

A final remark about the ACG class of functions, which are in some sense the widest general class of interest in analysis. Such functions need not have derivative almost everywhere, see [Go, p.101; S, p.224]; but if an ACG does have a derivative almost everywhere then we get yet another variant of the primitive problem.

UNIQUENESS THEOREM₄ *If F is continuous and ACG , and has a derivative almost everywhere with $F' = 0$ almost everywhere on $[a, b]$ then F is a constant function.*

A function f is then said to be Denjoy-Hinčin¹⁶, or \mathcal{DH} -, (or sometimes \mathcal{DK} -, see footnote 16), integrable, with F a \mathcal{DH} -primitive if there is an F , continuous, differentiable almost everywhere, ACG and with $F' = f$ almost everywhere; see [Go; S].

7.4 THE GENERALIZED RIEMANN INTEGRAL Another approach to the primitive problem was introduced much later by Kurzweil¹⁷ and Henstock¹⁸ independently;

¹⁶ А Я Хинчин, 1894–1959; also transliterated as Khintchine.

¹⁷ Jaroslav Kurzweil, 1926–.

¹⁸ Ralph Henstock, –.

see [Go, Chap.9; He; Ku]. Their method generalizes the Riemann approach to integration by replacing the uniformly small partitions of intervals used in the elementary definition, see 3.1, by partitions that are locally small.

Given a positive function δ , a tagged partition $\varpi = \{a_0, \dots, a_n; \xi_1, \dots, \xi_n\}$ is said to be δ -fine if

$$\xi_i - \delta(\xi_i) < x_{i-1} < \xi_i < x_i < \xi_i + \delta(\xi_i), \quad 1 \leq i \leq n. \quad (\delta)$$

Then we say that I is the *Henstock-Kurzweil integral*, \mathcal{HK} -integral, of f on $[a, b]$ if given any $\epsilon > 0$ there is a positive function $\delta(x)$ such that for all δ -fine partitions of the interval $[a, b]$ we have that $|S - I| < \epsilon$, S as in (1).

EXAMPLES (i) If f is the derivative of F , and $\epsilon > 0$ define the positive function δ by

$$|F(v) - F(u) - hF'(x)| = |F(v) - F(u) - (v - u)f(x)| < |v - u| \frac{\epsilon}{b - a},$$

if $|v - u| < \delta$, and let ϖ be a δ -fine partition, as above. Then,

$$\begin{aligned} \left| F(b) - F(a) - \sum_{i=1}^n f(\xi_i)(a_i - a_{i-1}) \right| &= \left| \sum_{i=1}^n (F(a_i) - F(a_{i-1}) - (a_i - a_{i-1})f(\xi_i)) \right| \\ &\leq \sum_{i=1}^n |F(a_i) - F(a_{i-1}) - (a_i - a_{i-1})f(\xi_i)| \\ &\leq \frac{\epsilon}{b - a} \sum_{i=1}^n (a_i - a_{i-1}) = \epsilon. \end{aligned}$$

So f is \mathcal{HK} -integrable and its primitive is a \mathcal{HK} -primitive.

Simple approaches to this integral can be found in [D-S; L-V; McL; McS].

7.5 RELATIONSHIPS BETWEEN THE VARIOUS SOLUTIONS The various definition of the integrals given above are very different and a natural question is — how are they related? They all extend the \mathcal{L} -integral, and in fact extend the \mathcal{HL} -integral. That is to say if a function is integrable in the Lebesgue sense then it is integrable in all the above senses and the integrals are equal; more, if the functions are non-negative or just bounded below, then if they are integrable in any of these senses they are also Lebesgue integrable. In addition these integrals have no Cauchy or Harnack extension— that is any such extension just give back the original integral; see Theorem 11(b). Finally all solve the primitive problem, that is Theorem 11 (c) holds with each of these integrals replacing the \mathcal{D}^* -integral

Surprisingly all these integrals are equivalent.

THE HAKE-LOOMAN-ALEKSANDROV THEOREM *If a function is integrable by the method of Luzin, or \mathcal{P} -, \mathcal{D}^* - or \mathcal{HK} -integrable then it is integrable by all these methods, and the integrals are equal.*

By far the most difficult part of this result is the equivalence of the Perron and restricted Denjoy integral, a problem solved by the three mathematicians H Hake, H Looman and Aleksandrov¹⁹ about ten years after Denjoy's initial paper.

The theorem says that the various integrals defined to solve the classical primitive problem are just different ways of looking at the same integral. This need not have been the case; as we see elsewhere the various integrals involved in the solution to the classical coefficient problem are not equivalent; see [Bu2].

Details of these integrals can be found in the classical book of Saks, [S. pp.186–259], as well as in the recent book by Gordon,[Go]; see also [Br; N vol.II; P].

7.6 THE POWER OF THE SOLUTION The classical primitive problem is well posed because, if F' is zero everywhere then F is a constant. There are many generalizations of this result and we can ask whether the methods discussed will solve these generalizations.

Perhaps the most elementary result is the following, see [B, pp.19–21].

UNIQUENESS THEOREM₅ *If F is continuous and $F'_+ = 0$ nearly everywhere then F is a constant function.*

So if f is nearly everywhere the right derivative of a continuous function we can ask for its primitive just as in the classical primitive problem. It can be shown that f is \mathcal{D}^* -integrable, and that the primitive is the \mathcal{D}^* -primitive.

Clearly the same problem, with same solution exists for the left derivative.

In this problem the requirement of continuity is essential, it cannot even be replaced by right continuity²⁰.

EXAMPLES (i) Define

$$F(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1/2, \\ \alpha, & \text{if } x = 1/2, \\ \beta, & \text{if } 1/2 < x \leq 1, \end{cases}$$

This example shows that we cannot allow one exceptional point in the classical primitive problem, since for all choices of α and β , $F'(x) = 0, x \neq 1/2$. In addition if $\alpha = \beta$ F is right continuous, so justifies the above remark.

¹⁹ П С Александров, 1896–1982; also transliterated as Alexandroff.

²⁰ The elaborate counter-example given in [B, p.27] seems to be false; see [Bus].

In addition we cannot allow an uncountable set exceptional set, even if it is of measure zero, as the following example shows; see [Go. pp.13–14; Ru, p.168].

EXAMPLES (ii) Let K be the **Cantor ternary set**. Now define F as follows: on the first removed open interval, of length $1/3$, F is put equal to $1/2$; on the two removed intervals of length $1/9$, put F equal to $1/4$ on the left interval and $3/4$ on the right interval; on the four removed intervals of length $1/27$ put F equal to $1/8, 3/8, 5/8, 7/8$, as the intervals are chosen from left to right; etc. In this way we define F off the set K ; on K it is defined by continuity. The resulting function F , called the *Cantor function*, is increasing, with $F(0) = 0, F(1) = 1$ and off K clearly $F' = 0$; that is $F' = 0$ almost everywhere.

This function is not of course constant ; in fact looking at Theorems 10 and 12 we deduce that F is not AC , or even ACG^* . In fact from the comments at the end of 6.3 it is not ACG either.

To obtain a correct extension of the primitive problem if uncountable exceptional sets are allowed we must either, as in Theorems 10 and 12 restrict the class of primitives even more than above, to AC or ACG^* functions, or have some other knowledge as in the following theorem; see [Br, p.120; K-K, pp.102–103; S, pp.205–207].

UNIQUENESS THEOREM₆ *If F is continuous and differentiable nearly everywhere with $F' = 0$ almost everywhere then F is a constant function.*

So if f is almost everywhere the derivative of a continuous nearly everywhere differentiable function we can ask for its primitive just as in the classical primitive problem. It can be shown that f is \mathcal{D}^* -integrable, and that the primitive is the \mathcal{D}^* -primitive; see[Go, p.108].

There are many other variants of the uniqueness theorem to be found in the various references. The discussion above shows the power of the integrals developed in that they are able to handle most of the problems that arise. The simplest way is to show that under the conditions given by a uniqueness theorem the primitive will be ACG^* , when the \mathcal{D}^* -integral will give the solution.

A rather surprising result is that recently the \mathcal{D}^* -integral has been shown to be too powerful. It is natural to ask if there is a minimal integral that will generalize the \mathcal{L} -integral and integrate all derivatives; that is this integral will do just that and no less general integral will. Surprisingly there is such an integral and it is not the \mathcal{D}^* -integral. its construction is based on an ingenious extension of the $\mathcal{H} - \mathcal{K}$ -integral. If (δ) is replaced by

$$\xi_i - \delta(\xi_i) < x_{i-1} < x_i < \xi_i + \delta(\xi_i), \quad 1 \leq i \leq n. \quad (\delta^*);$$

that is if we do not require the tag to lie in the sub-interval that it tags but just close by, then the integral defined, often called the *McShane integral*, is just the \mathcal{L} -integral; see [L-V p,127], [McS]. If in the definition of the integral the distance of the tag from the subinterval is restricted by requiring that $\sum_{i=1}^n \text{dist}(\xi_i, [x_{i-1}, x_i]) < 1/\epsilon$ we get an integral, called the \mathcal{C} -integral, that is strictly more general than the \mathcal{L} -integral and strictly less general than the \mathcal{D}^* -integral and is the minimal integral; see [B-D-P].

8 Appendix

Everything we have to say will occur in a closed bounded, compact, interval that may or may not be specified as $[a, b]$.

The *oscillation of a function f on the interval $[a, b]$* is

$$\omega(f; [a, b]) = \sup_{a \leq x, y \leq b} |f(y) - f(x)| = \sup_{a \leq x \leq b} f(x) - \inf_{a \leq x \leq b} f(x).$$

A property that holds except on a countable set is said to hold *nearly everywhere*.

8.1 OPEN AND CLOSED SETS An *open set*, G say, is either the empty set, or it is a union of a countable collection of non-overlapping open intervals, $I_n =]a_n, b_n[$, $n = 1, 2, \dots$ say; so $G = \bigcup_{n=1}^{\infty} I_n$

A *closed set* is either the empty set, or it is the complement, in the basic bounded closed interval, of an open set; so if G is as above $F = [a, b] \setminus G = [a, b] \setminus \bigcup_{n=1}^{\infty} I_n$ is a closed set. The closed intervals $[a_n, b_n]$, $n = 1, 2, \dots$, are called the *contiguous intervals* of F .

A point is said to be a *limit point of a set* if there is a sequence of distinct points of the set that converge to that point.

A closed set is distinguished by the fact that every limit point of the set is in the set. Not every point of a closed set need be a limit point—if that is the case then the set is called a *perfect set*.

A finite set has no limit points. An infinite set, in $[a, b]$, must have at least one limit point by the Bolzano-Weierstrass Theorem^{21,22}; see 8.3.

EXAMPLES (i) If $[a, b] = [0, 1]$ and $S = \{1, 1/2, 1/3, \dots, 1/n, \dots, 0\}$ then the only limit point is 0; its contiguous intervals are $[1/(n+1), 1/n]$, $n = 1, 2, \dots$

(ii) If $S = [a, b]$ then S is closed and every point is a limit point, so S is perfect.

Starting with a non-empty closed set $S = S^{(0)}$ say, let the set of its limit points be $S' = S^{(1)}$; this set is called the *first derived set of S* . It is also a closed set, see [R,

²¹ Bernard Placidus Johann Nepomuk Bolzano, 1781–1848.

²² Karl Wilhem Theodor Weierstraß, 1815–1897.

p. 55; *N vol.I, p. 37*], and is of course a subset of S . Further it is a proper subset unless S is perfect.

Unless S' is empty, or equivalently S is finite, this process can be repeated by taking all the limit points of $S^{(1)}$, getting the *second derived set of S* , $S^{(2)}$, a subset of $S^{(1)}$. Thus we can define a sequence $S^{(0)}, S^{(1)}, S^{(2)}, \dots$ of derived sets of higher and higher order.

There are three possibilities:

- (i) at some stage the *derived set of order n* , $S^{(n)}$, is empty, equivalently $S^{(n-1)}$ is finite; when of course the set S is countable;
- (ii) at some stage $S^{(n)}$ is not empty but is not a proper subset of $S^{(n-1)}$; that is $S^{(n-1)} = S^{(n)} \neq \emptyset$. So $S^{(n-1)}$ is a perfect set, all its points are limit points, and so S is not countable;
- (iii) neither of these happens, when not only do we get a different $S^{(n)}$ for all n , but in addition $S^{(\omega)} = \bigcap_{n=0}^{\infty} S^{(n)}$ is not empty, by the Cantor Intersection Theorem, see 8.3; this can happen both when S is countable, and when S is uncountable;

EXAMPLES (iii) If S is as in Example (i) $S^{(1)} = \{0\}$, $S^{(2)} = \emptyset$.

(iv) If $S = [1, 2] \cup S_1$, where S_1 is the set in Example (i), then $S^{(1)} = \{0\} \cup [1, 2]$, and $S^{(2)} = S^{(3)} = \dots = [1, 2]$.

(v) Let $a_n, n = 1, 2, \dots$ be any strictly decreasing sequence in $[0, 1]$ with limit 0 and put $A_1 = \{0, a_1, a_2, \dots\}$. Now on left-hand half of each $[a_{n+1}, a_n]$ put a copy of A_1 , and call the new set A_2 . Repeat this on each interval between two elements of A_2 . Keep on doing this and call the final set, the union of all this procedure, S . Then case (iii) above occurs, $S^{(n)}$ exist, and is non-empty, for all n , and is a proper subset of $S^{(n-1)}$; $S^{(\omega)} = \{0\}$. Clearly S is countable but $S \cup [1, 2]$ is uncountable, and this illustrates the last remark in case (iii).

Suppose then S is such that case (iii) holds. Then we can start the process all over again starting with $\tilde{S} = S^{(\omega)}$ to get $\tilde{S}' = S^{(\omega+1)}, S^{(\omega+2)} \dots$. This new sequence can exhibit all the three possibilities mentioned above. If again case (iii) occurs then the process can be repeated starting with $S^{(2\omega)} = \bigcap_{n=1}^{\infty} S^{(\omega+n)}$.

EXAMPLES (vi) If S is Examples (v) $S^{(\omega+1)}$ is empty.

(vii) If S is Examples (v) together with $[1, 2]$, then $S^{(\omega+1)} = \dots = [1, 2]$.

It is a very important result that sooner or later either case (i) or (ii) must arise; see [*Ho vol.I, pp.124–125*]. This is an example of the important Cantor-Baire Stationary Principle; see 8.3.

If the above procedure terminates with case (i) the original set was countable, and is called a *reducible* set. Otherwise the set was uncountable and is said to be *irreducible*; the resulting perfect set is called the *perfect kernel*, or *nucleus*, of the original closed set.

8.2 MEASURE The *length of any interval* I with endpoints a and b is $|I| = |b - a|$. A set E is said to be of *measure zero* if: give any $\epsilon > 0$ there are intervals $I_n, n = 1, 2, \dots$ such that (i) $E \subseteq \bigcup_{n=1}^{\infty} I_n$, and (ii) $\sum_{n=1}^{\infty} |I_n| < \epsilon$.

The complement, in our basic closed bounded interval, of a set of measure zero will be said to be of *full measure*.

A property that holds on a set of full measure is said to hold *almost everywhere*.

EXAMPLES (i) Any countable set $\{c_1, c_2, \dots\}$ has measure zero as can be seen by putting c_n inside $I_n =]c_n - \epsilon 2^{-n-1}, c_n + \epsilon 2^{-n-1}[$, $n = 1, 2, \dots$. In particular the empty set, any finite set, and the set of rationals \mathbb{Q} is of measure zero.

(ii) The set of irrationals is of full measure.

The *measure of the open set* $G = \bigcup_{n=1}^{\infty} I_n$ is $|G| = \sum_{n=1}^{\infty} |I_n|$

The *measure of the closed set* $F = [a, b] \setminus G$, is $|F| = b - a - |G|$.

EXAMPLES (ii) If the open set G is of full measure then the closed set F is of measure zero. This is easily checked since for every m , F is contained in the finite collection of closed intervals that make up the set $[a, b] \setminus \bigcup_{n=1}^m I_n$.

It is important to be able to exhibit a closed set of every measure from zero to full; see [K. pp.84–85]. The extremes are easy: finite sets are of zero measure, and of course $[a, b]$ is itself of full measure. These examples are not very useful and in the case of zero measure we can do better. There are many ways of constructing closed sets of a given measure but we will do it in a systematic and simple way.

Given $I_0 = [a, b]$ put $\mu_0 = |I_0| = b - a$ and suppose given a sequence $\underline{\epsilon}$ of positive real numbers ϵ_n , with $0 < \epsilon_n < 1$, $n = 1, 2, \dots$

Remove from I_0 the central open interval of length $\epsilon_1 \mu_0$, leaving two symmetrical closed intervals I_{11}, I_{12} both of the same length, μ_1 say; and clearly the total length of the two closed intervals is $2\mu_1 = \mu_0 - \epsilon_1 \mu_0 = \mu_0(1 - \epsilon_1) = (b - a)(1 - \epsilon_1)$.

Remove from each of these closed intervals, I_{11} and I_{12} , the central open interval of length $\epsilon_2 \mu_1$, leaving four symmetrically situated closed intervals I_{2i} , $1 \leq i \leq 4$, each of length μ_2 ; clearly $2\mu_2 = \mu_1 - \epsilon_2 \mu_1 = \mu_1(1 - \epsilon_2)$ so the total length of the four closed intervals is $4\mu_2 = 2\mu_1(1 - \epsilon_2) = \mu_0(1 - \epsilon_1)(1 - \epsilon_2) = (b - a)(1 - \epsilon_1)(1 - \epsilon_2)$.

At the n th stage we remove 2^{n-1} central open intervals each of length $\epsilon_n \mu_{n-1}$ leaving 2^n intervals I_{ni} , $1 \leq i \leq 2^n$ of length μ_n . The total length of the closed intervals remaining at the n th stage is

$$2^n \mu_n = \mu_0 \prod_{k=1}^n (1 - \epsilon_k) = (b - a) \prod_{k=1}^n (1 - \epsilon_k),$$

and so the length of the removed open intervals is $(b - a) (1 - \prod_{k=1}^n (1 - \epsilon_k))$.

Let then G be the open set that consists of the complete sequence of open intervals removed in this manner; and put K equal to the closed set that is the intersection

of all the remaining closed intervals obtained at each stage. Then $K = [a, b] \setminus G$ is not empty, by the Cantor Intersection Principle, see 8.3, and in any case is easily seen to contain at least the end points of all the removed open intervals. This set K is often called a *generalised Cantor set*, or a *Cantor-like set*; see [B-B-T, p.28]. From the above construction we have that:

$$|G| = (b - a) \left(1 - \prod_{k=1}^{\infty} (1 - \varepsilon_k) \right), \quad |K| = (b - a) \prod_{k=1}^{\infty} (1 - \varepsilon_k).$$

Now from the elementary theory of infinite products, see for instance [Kn, pp.218–229], $\prod_{k=1}^{\infty} (1 - \varepsilon_k)$ converges if and only if $\sum_{k=1}^{\infty} \varepsilon_k$ converges, while the product diverges to 0 if $\sum_{k=1}^{\infty} \varepsilon_k$ diverges.

Hence: if $\sum_{k=1}^{\infty} \varepsilon_k < \infty$ then $0 < |K| < b - a$; while if $\sum_{k=1}^{\infty} \varepsilon_k = \infty$ then $|K| = 0$.

EXAMPLES (i) If $\varepsilon_n = 1/3$, $n = 1, 2, \dots$, $a = 0, b = 1$, then obviously $\sum_{k=1}^{\infty} \varepsilon_k = \infty$; the above set, which is of zero measure, is known as the *Cantor ternary set*, or often just *the Cantor set*.

(ii) If $0 < \theta < \pi$, $\varepsilon_n = \theta^2/(n^2\pi^2)$, $n = 1, 2, \dots$ then $\sum_{k=1}^{\infty} \varepsilon_k < \infty$, see [Kn, p.221], and so K has positive measure. In this case the value of the infinite product is known to be $\sin\theta/\theta$, [Ru, p.310]. Since this last function takes every value between 0 and 1 we see that by the right choice of θ , K can have every measure greater than zero and less than $(b - a)$

It is worth noting some other properties of the Cantor sets:

- (a) K is a perfect set;
- (b) K is uncountable;
- (c) in every neighbourhood of a point of K can be found an interval of G ; this is expressed by saying that K is *nowhere-dense*;
- (d) when K has positive measure so does G and further they have a kind of fractal property; in every neighbourhood of every point of K there is a part of K of positive measure, and the same is true for G ; they are said to be *thick-in-themselves*.

8.3 SOME BASIC THEOREMS The following results are well known but will be referred to by name and are listed here, in the forms needed, for convenience.

BOLZANO-WEIERSTRASS THEOREM [R, p.53; N vol.I, pp.35–36]. *Every bounded closed set contains at least one limit point.*

CANTOR'S INTERSECTION THEOREM [B-B-T, pp.8–10; R, p.64]. *If the intersection of a decreasing sequence of bounded closed sets is empty then one of the sets in the sequence is itself empty*

In particular this shows that if none of the sets of the decreasing sequence is empty then their intersection cannot be empty.

CANTOR-BAIRE STATIONARY PRINCIPLE [Br, p.55; N vol.II, p.145]. *In the construction of a decreasing family of bounded closed sets one must after a countable number of steps arrive at a point where all the sets in the construction are the same.*

In particular if the family is strictly decreasing this means that after a countable number of steps the members of the family must all be empty.

8.4 DISCONTINUOUS DERIVATIVES A derivative need not be continuous, and it need not be bounded as the following standard example shows,

EXAMPLES (i) If $\alpha > 0$ and $\beta > 0$ define,

$$\phi_0(x) = \begin{cases} x^\alpha \sin x^{-\beta}, & \text{if } 0 < x \leq 1, \\ 0, & x = 0, \end{cases}$$

Then ϕ_0 is continuous, and if $\alpha > 1$ is differentiable with

$$\phi'_0(x) = \begin{cases} \alpha x^{\alpha-1} \sin x^{-\beta} - \beta x^{\alpha-\beta-1} \cos x^{-\beta}, & \text{if } 0 < x \leq 1, \\ 0, & x = 0. \end{cases}$$

So ϕ'_0 is continuous at the origin if $\alpha > \beta + 1$, and is not continuous at the origin if $1 < \alpha \leq \beta + 1$; it is unbounded there if $1 < \alpha < \beta + 1$. In addition if $1 < \alpha \leq \beta$ then ϕ'_0 is not \mathcal{L} -integrable in any interval that contains the origin; see [Br, p.52; Bul]; the standard example has $\alpha = \beta = 2$; see [K, pp.135–136].

This example can be elaborated by a standard process to produce a continuous function whose derivative has a dense set of discontinuities.

EXAMPLES (ii) Simple modifications of ϕ_0 will give a function defined on $[a, b]$ and which has the same characteristics at both a and b as ϕ_0 has at the origin:

$$\phi_{a,b}(x) = \begin{cases} \left(\frac{(x-a)(b-x)}{b-a} \right)^\alpha \sin \left(\frac{(x-a)(b-x)}{b-a} \right)^{-\beta}, & \text{if } a < x < b, \\ 0, & \text{if } x = a, \text{ or } x = b. \end{cases}$$

It is worth noting that $|\phi_{a,b}| \leq ((b-a)/4)^\alpha$

(iii) If ϕ_0 is as in Examples (i) and $r \in \mathbb{R}$ write $\phi_r(x) = \phi_0(x-r)$ with α, β depending on r . Let $C = \{c_1, c_2, \dots\}$, be any countable set, possibly dense such as the rationals, and put

$$\phi(x) = \sum_{n=1}^{\infty} \frac{\phi_{c_n}(x)}{n!}, \quad x \in \mathbb{R}.$$

Then we easily see that if $\alpha_{c_n} > 1, n = 1, 2, \dots$, ϕ is continuous and differentiable with

$$\phi'(x) = \sum_{n=1}^{\infty} \frac{\phi'_{c_n}(x)}{n!}. \quad x \in \mathbb{R};$$

further ϕ' is continuous except at the points of C where, at c_n , ϕ' has the same discontinuity as ϕ'_{c_n} .

In this way the set of points of discontinuity of a derivative can be countable, and can also be dense as we can take as the countable set to be the set of rationals; [Br, p. 34].

For further developments it is necessary to give a more elaborate example using the generalised Cantor set of 2.1.

EXAMPLES (iii) Now if K is a generalized Cantor set, as in 8.2, with contiguous intervals $I_n = [a_n, b_n]$, $n \in \mathbb{R}^*$ define, using Examples (ii) above, $\phi_K : [a, b] \rightarrow \mathbb{R}$ as follows

$$\phi_K(x) = \begin{cases} \phi_{a_n, b_n}(x), & x \in I_n, n = 1, 2, \dots; \\ 0, & x \in K. \end{cases}$$

Here the real numbers α, β are subject to the same conditions as Examples (i), (ii); in particular we always assume that α, β are positive and so ϕ_K is always continuous. It follows that ϕ_K exhibits at each point of K the character that ϕ_0 exhibited at the origin, namely:

- if $\alpha > 1$ then ϕ_K is differentiable ;
- if $\alpha > \beta + 1$ then ϕ'_K is continuous ;
- if $1 < \alpha \leq \beta + 1$ then ϕ'_K is not continuous at any point in K ;
- if $1 < \alpha < \beta + 1$ then every point of K is a point of unboundedness of ϕ'_K ;
- if $1 < \alpha \leq \beta$ then ϕ'_K is not \mathcal{L} -integrable in any neighbourhood of any point of K .

This example is due to Volterra²³; [Ho, vol.1, pp.490–491; J, pp.148–149]. It is to be noted from 8.2 that K can be so chosen that both it and its complement, G , are quite thick; both sets being in thick-in-themselves.

EXAMPLES (iv) Now if $n = 2, 3, \dots$ let K_n denote a generalized Cantor set with measure greater than $(1 - \frac{1}{n})(b - a)$; further let ϕ_n be the ϕ_{K_n} constructed as in Examples (iii), the numbers α, β being independent of n . Now define $\phi : [a, b] \rightarrow \mathbb{R}$ by

$$\phi(x) = \sum_{n=1}^{\infty} \frac{\phi_n(x)}{3^n}, \quad a \leq x \leq b.$$

Then ϕ is differentiable and its derivative has $E = \bigcup_{n=1}^{\infty} K_n$ as its points of discontinuity, unboundedness, or non \mathcal{L} -integrability, depending on the choices of α and β . Since now $|E| = b - a$, we have a derivative with a set of points of discontinuities, etc., that has full measure. A further discussion of this can be found in the references.

²³ Vito Volterra, 1860–1940.

Let us finish on a more positive note. Although a derivative need not be continuous, it is *Darboux Baire -1*. That is: (a) it takes any value between any two assumed values— the intermediate value property of continuous functions; see [Br, p.5; R, p.164]; (b) it is the limit of a sequence of continuous functions since

$$f'(x) = \lim_{n \rightarrow \infty} \phi_n(x), \quad \text{where} \quad \phi_n(x) = n(f(x + n^{-1}) - f(x)),$$

and each ϕ_n is continuous since f is, being differentiable.

Such functions have lots of points of continuity, in fact there are whole intervals of continuity on every perfect set; [B-B-T, pp.22–23]. So that on every perfect set the points of discontinuity are nowhere-dense.

9 References

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²⁴ Engl. transl by L F Boron of И П Натансон, Теория Функций Вещественной Переменной

²⁵ И Н Песин, Развитие Понятия Интеграла.

²⁶ Take care with this translation, some usages are historical, rather than modern.